Existence and uniqueness of a positive solution to generalized nonlocal thermistor problems with fractional-order derivatives*

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Abstract

In this work we study a generalized nonlocal thermistor problem with fractional-order Riemann–Liouville derivative. Making use of fixed-point theory, we obtain existence and uniqueness of a positive solution.

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1 Introduction

Joule heating is generated by the resistance of materials to electrical current and is present in any electrical conductor operating at normal temperatures. The heating of such conductors has undesirable side effects. Problems dealing with the combined heat and current flows were considered in [5, 9, 11, 13], where various aspects of the so-called thermistor problem were analyzed. The mathematical model of the nonlocal steady thermistor problem has the form

$$\triangle u = \frac{\lambda f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}, \tag{1}$$

where $\triangle$ is the Laplacian with respect to the spacial variables. Such problems arise in many applications, for instance, in studying the heat transfer in a resistor device whose electrical conductivity f is strongly dependent on the temperature u. The equation (1) describes the diffusion

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of the temperature with the presence of a nonlocal term as a result of Joule effect. Constant λ is a dimensionless parameter, which can be identified with the square of the applied potential difference at the ends of the conductor. Function u represents the temperature generated by the electric current flowing through a conductor. For more description, we refer to [12, 21]. A deep discussion about the history of thermistors, and more detailed accounts of their advantages and applications to industry, can be found in [11, 13]. In [4] Antontsev and Chipot studied existence and regularity of weak solutions to the thermistor problem under the condition that the electrical conductivity $f(u)$ is bounded.

Fractional differential equations are a generalization of ordinary differential equations and integration to arbitrary noninteger orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. In recent years there has been a great deal of interest in fractional differential equations. They provide a powerful tool for modeling and solving various problems in various fields: physics, mechanics, engineering, electrochemistry, economics, visco-elasticity, feedback amplifiers, electrical circuits and fractional multipoles — see, for example, [6, 8, 10, 18–20] and references therein. Using fixed point theorems, like Shauder's fixed point theorem, and Banach's contraction mapping principle, many results of existence have been obtained to linear and nonlinear equations and, more recently, to fractional derivative equations. The interested reader can see [2, 3]. For a physical meaning to the initial conditions of fractional differential equations with Riemann–Liouville derivatives we refer to [7, 15, 16].

Our main concern in this paper is to prove existence and uniqueness of solution to a general fractional order nonlocal thermistor problem of the form

$$\begin{gathered} D^{2\alpha}u = \frac{\lambda f(u)}{(\int_0^T f(u)\,dx)^2} + h(t)\,, \quad t \in (0,T)\,, \\ I^{\beta}u(t)|_{t=0} = 0, \quad \forall \beta \in (0,1], \end{gathered} \tag{2}$$

under suitable conditions on f and h (see Theorem 3.2). We assume that T is a fixed positive real and $\alpha > 0$ a parameter describing the order of the fractional derivative. In the literature we may find a great number of definitions of fractional derivatives. In this paper, the fractional derivative is considered in the Riemann–Liouville sense. In the case $\alpha = 1$ and $h = 0$, the fractional equation (2) becomes the one-dimensional nonlocal steady state thermistor problem. The values of $0 < \alpha < \frac{1}{2}$ correspond to intermediate processes. We further prove the boundedness of u (see Theorem 3.3), which is of considerable importance from a practical and physical point of view: it is interesting to keep the temperature from exceeding some extremal values that may damage the conductor.

2 Preliminaries

In this section, we give some basic definitions and preliminary facts that are used further in the paper. Let $0 < \alpha < \frac{1}{2}$ and $X = (C([0,T]), \|\cdot\|)$, where $C([0,T])$ is the space of all continuous functions on $[0,T]$. For $x \in C([0,T])$, define the norm

$$\|x\| = \sup_{t\in[0,T]} \{e^{-Nt}|x(t)|\},$$

which is equivalent to the standard supremum norm for $f \in C([0,T])$. It is used in literature in many papers, see for example [1]. The use of this norm is technical and allow us to simplify the integral calculus. By $L^1([0,T],\mathbb{R})$ we denote the set of Lebesgue integrable functions on $[0,T]$. Throughout the text c denote constants which may change at each occurrence. As in [4], we consider that the electrical conductivity is bounded. We now assume the following assumptions:

(H1) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lipschitz continuous function with Lipschitz constant L_f such that $c_1 \leq f(u) \leq c_2$, with c_1, c_2 two positive constants;

(H2) h is continuous on $(0,T)$ with $h \in L^\infty(0,T)$.

Definition 2.1. The fractional (arbitrary) integral of order $\alpha \in \mathbb{R}^+$ of a function $f \in L_1[a,b]$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds,$$

where Γ is the gamma function (see, e.g., [14, 16, 17, 19]). For $a = 0$ we put $I^\alpha := I_0^\alpha$.

Remark 2.1. For $f, g \in L_1[a,b]$ one has

$$I_a^\alpha(f(t) + g(t)) = I_a^\alpha f(t) + I_a^\alpha g(t).$$

Note also that $I^\alpha f(t) \in C(\mathbb{R}^+)$ for $f \in C(\mathbb{R}^+)$ and, moreover, $I^\alpha f(0) = 0$.

Definition 2.2 (see, e.g., [14, 16, 17, 19])**.** The Riemann–Liouville fractional (arbitrary) derivative of order $\alpha \in (n-1, n)$, $n \in \mathbb{N}$, of function f is defined by

$$D_a^\alpha f(t) = \frac{d^n}{dt^n} I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \quad t \in [a,b].$$

3 Main Results

Our main result asserts existence of a unique solution to (2) on $C(\mathbb{R}^+)$ of the form

$$\begin{aligned} u(t) &= I^{2\alpha}\left\{\frac{\lambda f(u)}{(\int_0^T f(u)\, dx)^2} + h(t)\right\} \\ &= \int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} \left\{\frac{\lambda f(u)}{(\int_0^T f(u)\, dx)^2} + h(s)\right\} ds\,. \end{aligned} \tag{3}$$

3.1 Existence and Uniqueness

We begin by proving the equivalence between (2) and (3) on the space $C(\mathbb{R}^+)$. This restriction of the space of functions allows to exclude from the proof a stationary function with Riemann-Liouville derivative of order 2α equal to $d \cdot t^{2\alpha-1}$, $d \in \mathbb{R}$, which belongs to the space $C_{1-2\alpha}[0,T]$ of continuous weighted functions.

Lemma 3.1. *Suppose that* $\alpha \in (0, \frac{1}{2})$. *Then the nonlocal problem* (2) *is equivalent to the integral equation* (3) *on the space* $C(\mathbb{R}^+)$.

Proof. First we prove that (2) implies (3). For $t > 0$ equation (2) can be written as

$$\frac{d}{dt} I^{1-2\alpha} u(t) = \frac{\lambda f(u)}{\left(\int_0^T f(u)\, dx\right)^2} + h(t).$$

Integrating both sides of the above equation, we obtain

$$I^{1-2\alpha} u(t) - I^{1-2\alpha} u(t)|_{t=0} = \int_0^t \left\{\frac{\lambda f(u)}{\left(\int_0^T f(u)\, dx\right)^2} + h(s)\right\} ds.$$

Since $0 < 1 - 2\alpha < 1$,

$$I^{1-2\alpha} u(t) = \int_0^t \left\{\frac{\lambda f(u)}{\left(\int_0^T f(u)\, dx\right)^2} + h(s)\right\} ds\,.$$

Applying the operator $I^{2\alpha}$ to both sides, we get

$$Iu(t) = I^{2\alpha+1}\left\{\frac{\lambda f(u)}{\left(\int_0^T f(u)\,dx\right)^2} + h(t)\right\}.$$

Differentiating both sides,

$$u(t) = I^{2\alpha}\left\{\frac{\lambda f(u)}{\left(\int_0^T f(u)\,dx\right)^2} + h(t)\right\}.$$

Let us now prove that (3) implies (2). Since $u \in C(I)$ and $I^{1-2\alpha}u(t) \in C(I)$, applying the operator $I^{1-2\alpha}$ to both sides of (3) one obtains

$$\begin{aligned} I^{1-2\alpha}u(t) &= I^{1-2\alpha}I^{2\alpha}\left\{\frac{\lambda f(u)}{\left(\int_0^T f(u)\,dx\right)^2} + h(t)\right\} \\ &= I\left\{\frac{\lambda f(u)}{\left(\int_0^T f(u)\,dx\right)^2} + h(t)\right\}. \end{aligned}$$

Differentiating both sides of the above equality,

$$DI^{1-2\alpha}u(t) = DI\left\{\frac{\lambda f(u)}{\left(\int_0^T f(u)\,dx\right)^2} + h(t)\right\}.$$

Then,

$$D^{2\alpha}u(t) = \frac{\lambda f(u)}{\left(\int_0^T f(u)\,dx\right)^2} + h(t), \quad t > 0.$$

□

Theorem 3.2. *Let f and h satisfy hypotheses $(H1)$ and $(H2)$. Then there exists a unique solution $u \in X$ of (2) for all $0 < \lambda < \frac{N^{2\alpha}}{L_f\left(\frac{1}{(c_1T)^2} + \frac{2c_2^2T}{(c_1T)^4}e^{NT}\right)}$.*

Proof. Let $F : X \to X$ be defined by

$$Fu = I^{2\alpha}\left\{\frac{\lambda f(u)}{\left(\int_0^T f(u)\,dx\right)^2} + h(t)\right\}.$$

Then,

$$\begin{aligned} |Fu - Fv| &= \left|I^{2\alpha}\left\{\frac{\lambda f(u)}{\left(\int_0^T f(u)\,dx\right)^2} - \frac{\lambda f(v)}{\left(\int_0^T f(v)\,dx\right)^2}\right\}\right| \\ &= \left|I^{2\alpha}\left\{\frac{\lambda}{\left(\int_0^T f(u)\,dx\right)^2}(f(u)-f(v)) + \lambda f(v)\left(\frac{1}{\left(\int_0^T f(u)\,dx\right)^2} - \frac{1}{\left(\int_0^T f(v)\,dx\right)^2}\right)\right\}\right| \\ &\leq \left|I^{2\alpha}\left\{\frac{\lambda}{\left(\int_0^T f(u)\,dx\right)^2}(f(u)-f(v))\right\}\right| + \left|I^{2\alpha}\left\{\lambda f(v)\left(\frac{1}{\left(\int_0^T f(u)\,dx\right)^2} - \frac{1}{\left(\int_0^T f(v)\,dx\right)^2}\right)\right\}\right|. \end{aligned} \tag{4}$$

We estimate each term on the right hand side of (4) separately. Using then the fact that f is Lipshitzian, we have

$$\begin{aligned}\left| I^{2\alpha}\left\{\frac{\lambda}{\left(\int_0^T f(u)\,dx\right)^2}(f(u)-f(v))\right\}\right| &\le \frac{1}{(c_1T)^2}\lambda I^{2\alpha}\{|f(u)-f(v)|\} \\ &\le \frac{1}{(c_1T)^2}\lambda L_f I^{2\alpha}\{|u-v|\} \\ &= \frac{1}{(c_1T)^2}\lambda L_f \int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}|u(s)-v(s)|ds.\end{aligned} \tag{5}$$

Since

$$\int_0^{Nt}\frac{r^{2\alpha-1}}{\Gamma(2\alpha)}e^{-r}dr \le \frac{1}{\Gamma(2\alpha)}\int_0^{+\infty} r^{2\alpha-1}e^{-r}dr = \frac{\Gamma(2\alpha)}{\Gamma(2\alpha)} = 1,$$

it follows from (5) that

$$\begin{aligned} e^{-Nt}&\left| I^{2\alpha}\left\{\frac{\lambda}{(\int_0^T f(u)\,dx)^2}(f(u)-f(v))\right\}\right| \\ &\le \frac{1}{(c_1T)^2}\lambda L_f e^{-Nt}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}|u(s)-v(s)|ds \\ &\le \frac{1}{(c_1T)^2}\lambda L_f \int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}e^{-Ns}|u(s)-v(s)|ds \\ &\le \frac{1}{(c_1T)^2}\lambda L_f \sup_t\{e^{-Nt}|u(t)-v(t)|\}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}ds \\ &\le \frac{1}{(c_1T)^2}\lambda L_f\|u-v\|\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}ds \\ &\le \frac{1}{(c_1T)^2}\lambda L_f\|u-v\|\frac{1}{N^{2\alpha}}\int_0^{Nt}\frac{r^{2\alpha-1}}{\Gamma(2\alpha)}e^{-r}dr \le \frac{\frac{1}{(c_1T)^2}\lambda L_f}{N^{2\alpha}}\|u-v\|.\end{aligned}$$

On the other hand, similar arguments as above yield to

$$\begin{aligned}&\left| I^{2\alpha}\left\{\lambda f(v)\left(\frac{1}{\left(\int_0^T f(u)\,dx\right)^2}-\frac{1}{\left(\int_0^T f(v)\,dx\right)^2}\right)\right\}\right| \\ &= \left| I^{2\alpha}\left\{\frac{\lambda f(v)}{\left(\int_0^T f(u)\,dx\right)^2\left(\int_0^T f(v)\,dx\right)^2}\left(\left(\int_0^T f(u)\,dx\right)^2-\left(\int_0^T f(v)\,dx\right)^2\right)\right\}\right| \\ &\le \frac{c_2}{(c_1T)^4}\lambda\left| I^{2\alpha}\left\{\left(\int_0^T f(u)\,dx\right)^2-\left(\int_0^T f(v)\,dx\right)^2\right\}\right| \\ &\le \frac{c_2}{(c_1T)^4}\lambda\left| I^{2\alpha}\left\{\left(\int_0^T (f(u)-f(v))\,dx\right)\left(\int_0^T (f(u)+f(v))\,dx\right)\right\}\right| \\ &\le \frac{2c_2^2T}{(c_1T)^4}\lambda I^{2\alpha}\left\{\int_0^T |f(u)-f(v)|\,dx\right\} \\ &\le \frac{2c_2^2T}{(c_1T)^4}\lambda L_f I^{2\alpha}\left\{\int_0^T |u(x)-v(x)|\,dx\right\} \\ &\le \frac{2c_2^2T}{(c_1T)^4}\lambda L_f\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}\left(\int_0^T |u(x)-v(x)|\,dx\right)ds.\end{aligned} \tag{6}$$

Then,

$$
\begin{aligned}
&e^{-Nt}\left|I^{2\alpha}\left\{\lambda f(v)\left(\frac{1}{(\int_0^T f(u)\,dx)^2}-\frac{1}{\left(\int_0^T f(v)\,dx\right)^2}\right)\right\}\right| \\
&\le \frac{2c_2^2T}{(c_1T)^4}\lambda L_f\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}\left(\int_0^T e^{-N(t-x)}e^{-Nx}|u(x)-v(x)|\,dx\right)ds \\
&\le \frac{2c_2^2T}{(c_1T)^4}\lambda L_f\sup_t\{e^{-Nt}|u(t)-v(t)|\}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}\left(\int_0^T e^{-N(t-x)}\,dx\right)ds \\
&\le \frac{2c_2^2T}{(c_1T)^4}\lambda L_f\sup_t\left\{e^{-Nt}|u(t)-v(t)|\right\}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-Nt}\left(\frac{1}{N}(e^{NT}-1)\right)ds \\
&\le \frac{2c_2^2T}{(c_1T)^4}\lambda L_f\|u-v\|\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-Nt}\left(\frac{1}{N}(e^{NT}-1)\right)ds \\
&\le \frac{\frac{2c_2^2T}{(c_1T)^4}e^{NT}\lambda L_f}{N}\|u-v\|\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-Nt}ds \\
&\le \frac{\frac{2c_2^2T}{(c_1T)^4}e^{NT}\lambda L_f}{N}\|u-v\|\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}e^{-Ns}ds \\
&\le \frac{\frac{2c_2^2T}{(c_1T)^4}e^{NT}\lambda L_f}{N}\|u-v\|\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}ds \\
&\le \frac{\frac{2c_2^2T}{(c_1T)^4}e^{NT}\lambda L_f}{N^{2\alpha+1}}\|u-v\| \\
&\le \frac{\frac{2c_2^2T}{(c_1T)^4}e^{NT}\lambda L_f}{N^{2\alpha}}\|u-v\|.
\end{aligned}
\tag{7}
$$

Gathering (4)–(7), we get

$$e^{-Nt}|Fu-Fv|\le\left(\frac{1}{(c_1T)^2}+\frac{2c_2^2T}{(c_1T)^4}e^{NT}\right)\frac{\lambda L_f}{N^{2\alpha}}\|u-v\|.$$

Then, we have

$$\|Fu-Fv\|\le\left(\frac{1}{(c_1T)^2}+\frac{2c_2^2T}{(c_1T)^4}e^{NT}\right)\frac{\lambda L_f}{N^{2\alpha}}\|u-v\|.$$

Choosing $\lambda>0$ such that $\left(\frac{1}{(c_1T)^2}+\frac{2c_2^2T}{(c_1T)^4}e^{NT}\right)\frac{\lambda L_f}{N^{2\alpha}}<1$, the map $F:X\to X$ is a contraction and it has a fixed point $u=Fu$. Hence, there exists a unique $u\in X$ that is the solution to the integral equation (3). The result follows from Lemma 3.1. □

3.2 Boundedness

We now show that the condition that the electrical conductivity $f(u)$ is bounded (hypothesis (H1)) allows to assert boundedness of u.

Theorem 3.3. *Under hypotheses (H1) and (H2) and $\lambda>0$, if u is the solution of (3), then*

$$\|u\|\le\frac{\left(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty\right)}{N^{2\alpha}}e^{\frac{\lambda L_f}{(c_1TN^\alpha)^2}}.$$

Proof. One has

$$
\begin{aligned}
|u(t)| &\le I^{2\alpha}\left\{\frac{\lambda|f(u)|}{(\int_0^T f(u)\,dx)^2}+|h(t)|\right\}\\
&\le \frac{\lambda}{(c_1T)^2}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}|f(u(s))-f(0)|ds+\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}\left(|h(s)|+\frac{\lambda}{(c_1T)^2}f(0)\right)ds\\
&\le \frac{\lambda L_f}{(c_1T)^2}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}|u(s)|ds+\left(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty\right)\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}ds.
\end{aligned}
$$

Then,

$$
\begin{aligned}
e^{-Nt}|u(t)| &\le \frac{\lambda L_f}{(c_1T)^2}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-Nt}|u(s)|ds+(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-Nt}ds\\
&\le \frac{\lambda L_f}{(c_1T)^2}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}e^{-Ns}|u(s)|ds+(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}e^{-Ns}ds\\
&\le \frac{\lambda L_f}{(c_1T)^2}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}e^{-Ns}|u(s)|ds+(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}ds\\
&\le \frac{\lambda L_f}{(c_1T)^2}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}e^{-Ns}|u(s)|ds+\frac{(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)}{N^{2\alpha}}\int_0^{Nt}\frac{r^{2\alpha-1}}{\Gamma(2\alpha)}e^{-r}dr\\
&\le \frac{(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)}{N^{2\alpha}}+\frac{\lambda L_f}{(c_1T)^2}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}\left(e^{-Ns}|u(s)|\right)ds.
\end{aligned}
$$

Using Gronwall's lemma, we have

$$
\begin{aligned}
e^{-Nt}|u(t)| &\le \frac{(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)}{N^{2\alpha}}e^{\frac{\lambda L_f}{(c_1T)^2}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}e^{-N(t-s)}ds}\\
&\le \frac{(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)}{N^{2\alpha}}e^{\frac{\frac{\lambda L_f}{(c_1T)^2}}{N^{2\alpha}}\int_0^{Nt}\frac{r^{2\alpha-1}}{\Gamma(2\alpha)}e^{-r}dr}\\
&\le \frac{(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)}{N^{2\alpha}}e^{\frac{\frac{\lambda L_f}{(c_1T)^2}}{N^{2\alpha}}}.
\end{aligned}
$$

Then,

$$
\|u\| \le \frac{(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)}{N^{2\alpha}}e^{\frac{\frac{\lambda L_f}{(c_1T)^2}}{N^{2\alpha}}}=\frac{(\frac{\lambda}{(c_1T)^2}f(0)+h_\infty)}{N^{2\alpha}}e^{\frac{\lambda L_f}{(c_1TN^\alpha)^2}}
$$

and we conclude that u is bounded. □

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